AM≤GM

The Arithmetic/Geometric Mean Inequality (AM/GM) is often not seen until graduate school, but is a very powerful problem solving tool. It states that for n positive real numbers $a_1, a_2, \ldots, a_n$,

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}, \tag{1}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

(a) Prove AM/GM for $n = 2$; i.e., prove that for positive numbers a and b,

$$\sqrt{ab} \leq \frac{a+b}{2},$$

with equality iff $a = b$.

(b) Use AM/GM to show that

$$n! \leq \left(\frac{n+1}{2}\right)^n,$$

for all positive integers n. When do we have equality? (Hint: Replace each a_i in (1) by i.)

(c) Use AM/GM to show that for n positive real numbers $a_1, a_2, \ldots, a_n$,

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. This is called the Harmonic/Arithmetic Mean Inequality. (Hint: Replace each a_i in (1) by $1/a_i$.)