Fun with a limit

(a) Compute $\int_0^1 \ln x\, dx$.

(b) Write the integral in part (a) as a limit of right Riemann sums.

(c) Below is a (correct) proof of the fact that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1}.$$

Give reasons for *each* numbered step.

$$\begin{aligned}
\frac{\sqrt[n]{n!}}{n} &= \exp\left[\ln\left(\frac{\sqrt[n]{n!}}{n}\right)\right] && (1)\\
&= \exp\left[\frac{1}{n}\ln n! - \ln n\right] && (2)\\
&= \exp\left[\frac{1}{n}\sum_{k=1}^{n}\ln k - \ln n\right] && (3)\\
&= \exp\left[\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)\right] && (4)\\
&\to e^{-1}, \text{ as } n\to\infty. && (5)
\end{aligned}$$

Bonus: A classic result in analysis is that if (a_n) is a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l,$$

then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = l.$$

Use this result to give another proof of the fact that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1}.$$