An interesting function

We know that a differentiable function is always continuous, but is its *derivative* necessarily continuous? This example shows that the answer to this question is, unfortunately, **NO!**

Define

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} .$$

(a) Use the Squeeze Theorem to show that

$$\lim_{x \to 0} f(x) = 0.$$

Why does this prove that $f(x)$ is continuous everywhere?

(b) Find $f'(x)$ for $x \neq 0$.

(c) Use the limit definition of the derivative to find $f'(0)$. (Squeeze Theorem is useful again!)

(d) Show that $\lim_{x \to 0} f'(x)$ does not exist. Why does this prove that $f'(x)$ is **not** continuous?